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On instantons and zero modes of $\mathcal{N} = 1/2$ SYM theory

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Abstract: We study zero modes of $\mathcal{N} = 1/2$ supersymmetric Yang-Mills action in the background of instantons. In this background, because of a quartic antichiral fermionic term in the action, the fermionic solutions of the equations of motion are not in general zero modes of the action. Hence, when there are fermionic solutions, the action is no longer minimized by instantons. By deforming the instanton equation in the presence of fermions, we write down the zero modes equations. The solutions satisfy the equations of motion, and saturate the BPS bound. The deformed instanton equations imply that the finite action solutions have U(1) connections which are not flat anymore.

Keywords: Solitons Monopoles and Instantons, Supersymmetry and Duality.

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1. Introduction

The duality correspondence between supersymmetric gauge theories and matrix models [1, 2] has had many interesting implications. Having originated from string theory, the duality with matrix models can be extended even beyond the known supersymmetric gauge theories. One such generalization comes by studying the D-branes in the background of graviphoton field [3, 4, 5]. In this background, the spinor coordinates θ^α on the brane turn out to be nonanticommuting variables. And the effective theory on the brane comes out as an exotic supersymmetric gauge theory with nonanticommuting fermionic fields. In another approach, assuming that superspace coordinates do not anticommute, Seiberg [6] has shown that one can still construct a super Yang-Mills lagrangian which preserves only half of the supersymmetries. The renormalizability of this model has been studied in [7]–[13]. While the generalization to $\mathcal{N} = 2$, along with other interesting aspects of noncommutative superspace have further been explored in [14]–[20].

The analysis of (anti)instantons in $\mathcal{N} = 1/2$ SYM model parallels the one in ordinary $\mathcal{N} = 1$ SYM theory, though, in the former case the equation of motion for chiral fermions λ becomes more involved when $F^+ = 0$. In this background, there are antichiral $\overline{\lambda}$ zero modes, which deform the equation for chiral fermions so that λ cannot remain zero. Further, because of a quartic antichiral term in the action, the fermionic solutions to the eqautions of motion are not in general the zero modes of the action. This will have a further consequence that in the presence of fermions, instantons are not solutions to the equations of motion anymore. This can also be seen directly by noticing that in the presence of fermionic solutions (not the zero modes) the action has a value greater than the instanton charge, the difference being proportional to the deformation parameter $C^{\alpha\beta}$. In this note we would like to comment that in $\mathcal{N} = 1/2$ U(2) SYM model the instanton equation should be deformed as follows,

$$F^+_{\mu\nu} + \frac{i}{2} C_{\mu\nu} \overline{\lambda\lambda} = 0 \tag{1.1}$$

$$\not{D}\overline{\lambda} = 0 \,, \tag{1.2}$$

where $F^+_{\mu\nu}$ is the self-dual part of the field strength, and $C_{\mu\nu}$ being the deformation parameter. The above equations, as we will see shortly, are also the zero mode equation for $\overline{\lambda}$. The solutions of the above equations, like instantons and their corresponding zero modes say in $\mathcal{N}=1$ theory, saturate a BPS bound in each topological sector specified by the instanton number k. However, unlike instantons in the presence of fermionic solutions in $\mathcal{N}=1/2$ theory, they do satisfy the equations of motion. This happens partly because in this configuration $\lambda=0$ is still a solution.

In what follows, we will see that for these states only the U(1) part of the instanton equation is deformed and finite action solutions will have nonflat U(1) connection. The SU(2) instanton equation and the corresponding Dirac equation for the adjoint fermions, on the other hand, remain undeformed. This should be contrasted with the case of supersymmetric solutions (instantons) with fermions set to zero, where the finiteness of action requires one to consider only the flat U(1) connections. In the case of 't Hooft one-instanton solution and the corresponding fermionic zero modes, we obtain the solutions of the U(1) gauge fields. These, however, will have a zero U(1) instanton number and thus belong to the $k=-1$ topological sector. This is in agreement with the fact that there are no U(1) instantons except the flat ones, and that eqs. (1.1) and (1.2) are the zero modes equations.

2. Instantons and zero modes

Let us begin with assuming that superspace coordinates θ^α are not anticommuting, and instead the satisfy the following anticommutation relation

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}\,, \tag{2.1}$$

where $C^{\alpha\beta}$ is a constant and symmetric 2×2 matrix. This will deform the supersymmetry algebra with $\overline{Q}^2$ proportional to the deformation parameter $C^{\alpha\beta}$. Seiberg [6] has considered the above deformation in $\mathcal{N}=1$ supersymmetric model and has shown that half of the supersymmetries can be preserved. Indeed if $W^\alpha=(A_\mu,\lambda)$ denotes the usual $\mathcal{N}=1$ gauge super multiplet, then the lagrangian of this $\mathcal{N}=1/2$ model reads

$$\begin{aligned}\mathcal{L} = i\tau\int d^2\theta\,\mathrm{tr}\,W^\alpha W_\alpha - i\bar\tau\int d^2\bar\theta\,\mathrm{tr}\,\overline{W}^{\bar\alpha}\overline{W}_{\bar\alpha} + \\ +(i\tau - i\bar\tau)\left(-iC^{\mu\nu}\,\mathrm{tr}\,F_{\mu\nu}\overline{\lambda}\overline{\lambda} + \frac{|C|^2}{4}\,\mathrm{tr}(\overline{\lambda}\overline{\lambda})^2\right)\,,\end{aligned} \tag{2.2}$$

where

$$C^{\mu\nu} \equiv C^{\alpha\beta}\epsilon_{\beta\gamma}\sigma^{\mu\nu\ \gamma}_{\alpha}$$

is a constant and antisymmetric self-dual matrix, and $|C|^2 = C_{\mu\nu}C^{\mu\nu}$.

The above lagrangian is invariant under the following Q deformed supersymmetry transformations,

$$\begin{aligned}\delta\lambda &= i\epsilon D + \sigma^{\mu\nu}\epsilon\left(F_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\overline{\lambda}\overline{\lambda}\right)\\ \delta A_\mu &= -i\overline{\lambda}\bar\sigma_\mu\epsilon\\ \delta D &= -\epsilon\sigma^\mu D_\mu\overline{\lambda}\\ \delta\overline{\lambda} &= 0\,,\end{aligned}$$

whereas $\overline{Q}$ is broken. A supersymmetric state is invariant under the above supersymmetric transformations. Setting λ and $\overline{\lambda}$ to zero, with

$$F^+_{\mu\nu} = 0\,, \tag{2.3}$$

gives such a state preserving the whole unbroken supersymmetry. This is of course also a solution to the equations of motion. Moreover, in $\mathcal{N} = 1$ SYM theory, since in the background of instantons ($F^+ = 0$) there are antichiral zero modes with no chiral zero modes, instantons remain solutions to the equations of motion in the presence of fermionic solutions. But in $\mathcal{N} = 1/2$ SYM theory this is not the story.

As mentioned in Introduction, in the background of instantons, and because the action of $\mathcal{N} = 1/2$ SYM theory has a quartic antichiral fermionic term, the zero modes, in general, do not satisfy the equations of motion. Therefore, instantons will not remain solutions when there are fermions. To remedy this, we deform the instanton equation and show that they satisfy the equations of motion and like instantons have a finite action. The equations are

$$\begin{aligned} F^+_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\overline{\lambda\lambda} &= 0 \\ \not{D}\overline{\lambda} &= 0 \\ \lambda &= 0\,. \end{aligned} \tag{2.4}$$

It is easy to see that a solution to (2.4) is also a solution to the equations of motion. In fact, if we set $\lambda = 0$, the equation of motion for the gauge fields is satisfied:

$$\begin{aligned} D_\mu\left(F^{\mu\nu} + iC^{\mu\nu}\overline{\lambda\lambda}\right) &= D_\mu\left(F^{\mu\nu-} + F^{\mu\nu+} + \frac{i}{2}C^{\mu\nu}\overline{\lambda\lambda} + \frac{i}{2}C^{\mu\nu}\overline{\lambda\lambda}\right) \\ &= D_\mu\left(F^{\mu\nu-} + \frac{i}{2}C^{\mu\nu}\overline{\lambda\lambda}\right) \\ &= D_\mu\left(F^{\mu\nu+} + \frac{i}{2}C^{\mu\nu}\overline{\lambda\lambda}\right) = 0\,, \end{aligned} \tag{2.5}$$

where in the last equality we used the Bianchi identity

$$D_\mu(F^{\mu\nu+} - F^{\mu\nu-}) = 0\,, \tag{2.6}$$

with

$$F^\pm_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right). \tag{2.7}$$

The equations of motion for $\overline{\lambda}$ and λ read

$$\begin{aligned} \overline{\not{D}}\lambda &= -C^{\mu\nu}F^+_{\mu\nu}\overline{\lambda} - i\frac{|C|^2}{2}(\overline{\lambda\lambda})\overline{\lambda} \\ \not{D}\overline{\lambda} &= 0\,. \end{aligned} \tag{2.8}$$

which are also satisfied by solutions of (2.4).

Equivalently, one can check that the solutions to (2.4) are absolute minima of the action in each topological sector. To see this, let us again set $\lambda = 0$, and add to the action a topological term proportional to the instanton number

$$\begin{aligned} S + k &= \int d^4x \ \mathrm{tr} \left(\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + i C^{\mu\nu} F_{\mu\nu} \overline{\lambda\lambda} - \frac{|C|^2}{4} (\overline{\lambda\lambda})^2 + \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right) \\ &= \int d^4x \ \mathrm{tr} \left(F^{\mu\nu+} F^{+}_{\mu\nu} + i C^{\mu\nu} F_{\mu\nu} \overline{\lambda\lambda} - \frac{|C|^2}{4} (\overline{\lambda\lambda})^2 \right) \\ &= \int d^4x \ \mathrm{tr} \left(F^{+}_{\mu\nu} + \frac{i}{2} C_{\mu\nu} \overline{\lambda\lambda} \right)^2 \geq 0 \,, \end{aligned} \tag{2.9}$$

henec for configurations satisfying (2.4), the action is minimized and equal to $-k$. This also signals that fermions satifying (2.4) are the true zero modes of the action.

For U(1) gauge group, since $\overline{\lambda}$ is in the adjoint representation, the second equation of (2.4) reduces to

$$\partial\!\!\!/ \overline{\lambda} = 0 \tag{2.10}$$

which has no normalizable solution on $\mathbb{R}^4$. Therefore in the U(1) case, eqs. (2.4) reduce to the abelian instanton equations which are known to have no nontrivial solutions on $\mathbb{R}^4$ except the flat ones. However, for gauge groups of higher rank, there might be nontrivial solutions different than instantons. Let us hence consider the U(2) gauge group.

3. Analysis of zero mode equations

In this section we would like to analyse possible solutions to eqs. (2.4). We will see that in the background of fermionic solutions of Dirac equation, the U(1) part of the connection can no longer be flat. Specially, we find solutions in the presence of adjoint fermions in the background of 't Hooft instantons. The U(1) connections we find have a zero instanton number which is consistent with the fact that the deformed instanton equations are also the equations for zero modes.

To begin with, let $T^a = (T^i = \sigma^i/2, T^4 = 1)$, $i = 1, 2, 3$ denote the generators of SU(2) and U(1) groups respectively. Now, to isolate the SU(2) and U(1) parts of eqs. (2.4), we expand the quadratic term $\overline{\lambda\lambda}$ in that equation

$$\begin{aligned} \overline{\lambda\lambda} &= \overline{\lambda}^a \overline{\lambda}^b T^a T^b = \frac{1}{2} \overline{\lambda}^a \overline{\lambda}^b \left([T^a, T^b] + \{T^a, T^b\} \right) \\ &= \frac{1}{2} \overline{\lambda}^i \overline{\lambda}^j \left(i\epsilon_{ijk} T^k + \frac{\delta_{ij}}{2} \right) + 2 \overline{\lambda}^i \overline{\lambda}^4 \, T^i + \overline{\lambda}^4 \overline{\lambda}^4 \\ &= \frac{1}{4} \overline{\lambda}^i \overline{\lambda}^i + 2 \overline{\lambda}^i \overline{\lambda}^4 \, T^i + \overline{\lambda}^4 \overline{\lambda}^4 \,. \end{aligned} \tag{3.1}$$

So, for the SU(2) part we have

$$F^{+i}_{\mu\nu} + i C_{\mu\nu} \overline{\lambda}^i \overline{\lambda}^4 = 0 \tag{3.2}$$

$$D\!\!\!\!/ \, \overline{\lambda}^i = 0 \,, \tag{3.3}$$

while the U(1) part reads

$$F^{+4}_{\mu\nu} + \frac{i}{8} C_{\mu\nu} \overline{\lambda}^i \overline{\lambda}^i + \frac{i}{2} C_{\mu\nu} \overline{\lambda}^4 \overline{\lambda}^4 = 0 \tag{3.4}$$

$$\partial\!\!\!/ \overline{\lambda}^4 = 0 \,. \tag{3.5}$$

As said before, the last equation has no normalizable solution and thus we set $\overline{\lambda}^4 = 0$. This will reduce eqs. (3.2) and (3.3) to the ordinary SU(2) instanton equation and the corresponding Dirac equation for the adjoint antichiral fermions. Equation (3.4), however, reads

$$F^{+4}_{\mu\nu} + \frac{i}{8} C_{\mu\nu} \overline{\lambda}^i \overline{\lambda}^i = 0 \,, \tag{3.6}$$

where $\overline{\lambda}^i$ are solutions to (3.3). This is the equation we would like to further study. For this we consider the 't Hooft one instanton solution and the fermionic modes in the adjoint representation satisfying (3.3).

A solution to Dirac equation (3.3) can in general be written in terms of ADHM data [21, 22, 23]. In particular, in the background of one instantons one can write the explicit solutions [21, 24]. For the SU(2) case, we quote the result for three different normalized solutions

$$\mathrm{tr}(\overline{\lambda\lambda}) = \frac{2}{\pi^2} \rho^2 f(x)^3 \tag{3.7}$$

$$\mathrm{tr}(\overline{\lambda\lambda}) = \frac{6}{\pi^2} \rho^4 f(x)^4 \tag{3.8}$$

$$\mathrm{tr}(\overline{\lambda\lambda}) = \frac{3}{\pi^2} \rho^2 r^2 f(x)^4 \,, \tag{3.9}$$

where

$$f(x) = \frac{1}{r^2 + \rho^2} \,, \tag{3.10}$$

and $r = \sqrt{\sum_{i=1}^4 x_i^2}$. Here ρ is the instanton size, and we have set to zero the instanton position for simplicity. The analysis of eq. (3.6) becomes easier if we make use of the Corrigan's identities

$$\rho^2 f(x)^3 = -\frac{1}{8} \partial^\alpha \partial_\alpha f(x) \tag{3.11}$$

$$\rho^4 f(x)^4 = \frac{1}{24} \partial^\alpha \partial_\alpha \left(r^2 f(x)^2 - 2 f(x) \right) \tag{3.12}$$

$$r^2 \rho^2 f(x)^4 = -\frac{1}{24} \partial^\alpha \partial_\alpha \left(r^2 f(x)^2 + f(x) \right) \,. \tag{3.13}$$

Eq. (3.6) now reads

$$F^{+4}_{\mu\nu} = -\frac{i}{8} C_{\mu\nu} \partial^\alpha \partial_\alpha K(x) \,, \tag{3.14}$$

for $K(x)$ being any of the functions $\frac{2}{\pi^2} f(x)\,, \frac{6}{\pi^2}(r^2 f(x)^2 - 2f(x))$ or $\frac{3}{\pi^2}(r^2 f(x)^2 + f(x))$ appearing on the right hand sides of eqs. (3.11), (3.12), and (3.13), respectively. The solution to this equation — up to a gauge transformation — is then found to be

$$A_\mu(x) = \frac{i}{8} C_{\mu\nu} \partial_\nu K(x) \,, \tag{3.15}$$

which of course has a nonvanishing curvature. Therefore, what we have found is that in the presence of fermionic zero modes the U(1) part of the connection cannot remain flat. This is in contrast with the supersymmetric solutions to the field equations (instantons) where one sets the fermions to zero, and for having a finite action solution one has to restrict to the flat part of the U(1) connections.

For instantons of higher topological charge, as mentioned earlier, one can write the adjoint zero mode solutions in terms of the ADHM data. Interestingly, using the Corrigan's identity, it can be seen that the expression for $\mathrm{tr}(\overline{\lambda\lambda})$ is actually a total derivative for all topological charges k,

$$\mathrm{tr}(\overline{\lambda\lambda})_k = \partial^\alpha \partial_\alpha \mathcal{K}(x;k)\,, \tag{3.16}$$

for some function $\mathcal{K}(x;k)$ which in turn has an explicit expression in terms of ADHM data [21, 23]. Therefore, we conclude that the solutions to (3.6) for all values of k will have the general form of the solution we found for $k = -1$ in (3.15). Moreover, since $\mathcal{K}(x;k)$ has no singularity and at infinity goes like r^{-2}, the corresponding U(1) gauge fields will all have zero instanton number.

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